Using Generalized Lagrange Chebyshev collocation method in order to solve the shortest path problems

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ABSTRACT

ABSTRACT. A theorem of Hardy, Littlewood and Polya has been used to represent the variational form of shortest path problem (SPP). As consequence of theorem, the SPP is converted into optimal control problem. In this paper, simply Generalized Lagrange Chebyshev collocation method is applied for solving the SPP. We finally present our numerical findings that demonstrate the efficiency and applicability of the numerical scheme by considering an example.

INTRODUCTION

The SPP has been considered in several works for example see [1–3], the SPP is modelled as follows:

$$Min \qquad J = \int_{\alpha}^{\beta} \sqrt{1+s'^2(t)}dt, \tag{1.1}$$

$$\text{subject to} \quad F_1(t) \le s(t) \le F_2(t), \quad \forall_t \in [\alpha, \beta],$$
$$s(\alpha) = x_0, \quad s(\beta) = x_1.$$

For solving the SPP problem, unknown function is approximated based on Generalized Lagrangian Chebyshev polynomials and by utilizing Gauss-Chebyshev integration rule see [4], and collocate in zeros of shifted Chebyshev polynomials, the problem is converted to nonlinear programming problems (NLPs) that simply could be solved using *NLPSolve* command in MAPLE software.

Preliminaries

2.1. **Generalized Lagrange functions.** Suppose $w(x) = \prod_{i=0}^{N}(u(x)-u(x_i))$ and $u(x)$ is continuous and sufficiently differentiable function, the generalized Lagrange functions are defined as follows:

$$L_j^u(x) = \frac{w(x)}{(u-u_j)\partial_x w(x_j)} = \frac{u_j' w(x)}{(u-u_j)\partial_u w(x_j)} = k_j \frac{w(x)}{(u-u_j)}, \qquad j = 0, ..., N, \tag{2.1}$$

where $k_j = \frac{u_j'}{\partial_u w(x_j)}$. Furthermore, the generalized Lagrange functions are satisfied in the Kronecker delta property. For more details see [4].

2.2. **Generalized Lagrange Chebyshev functions.** Chebyshev polynomials are orthogonal in the interval $[-1, 1]$, and by introducing the change of variable $z = \frac{2}{\beta-\alpha}(x-\alpha) - 1$ the so-called shifted Chebyshev polynomials are defined that are orthogonal in the interval $[\alpha, \beta]$. In generalized Lagrange Chebyshev functions, $w(x)$ in Eq. (2.1) could be considered as follows:

$$w(x) = P_N(u(x))$$

and $x_j \quad j = 1...N+1$ are zeros of shifted Chebyshev polynomial of degree $N+1$. The assumed function $s(x)$ that is defined over the interval $[\alpha, \beta]$ may expanded as:

$$s(x) = \sum_{i=0}^{\infty} a_i L_i^u(x),$$

for more details see [4].

References

[1] E. Tohidi and OR. Navid Samadi, *Legendre spectral collocation method for approximating the solution of shortest path problems*, Int. J. Control. **3** (2015), no. 1, 62–68.

[2] K. Mamehrashi and S.A. Yousefi, *A numerical method for solving a nonlinear 2-D optimal control problem with the classical diffusion equation*, Int. J. Control. **90** (2016), no. 2, 298–306.

[3] M. Zamirian , M.H. Farahi and A.R. Nazemi, *An applicable method for solving the shortest path problems*, Appl. Math. Comput. **190** (2007), no. 2, 1479–1486.

[4] K. Parand, S. Latifi, M. M. Moayeri and M. Delkhosh, *Generalized Lagrange Jacobi Gauss-Lobatto (GLJGL) Collocation Method for Solving Linear and Nonlinear Fokker-Planck Equations*, Commun. Theor. Phys.**69** (2018), no. 5, 519–531.

Proposed Method

By substitution $s(x) \simeq s_N(x) = \sum_{i=0}^{N} a_i L_i^u(x)$ in Eq. (1.1) and using Gauss-Chebyshev integration rule and also collocate at zeros of shifted Chebyshev polynomial of degree $N-2$ in constraints, the problem is converted to following NLPs as follows:

$$Min \qquad J = \int_{\alpha}^{\beta} \sqrt{1+s'^2(t)}dt = \sum_{i=0}^{N} C_i G(a_i, ..., a_N),$$

$$\text{s.t.} \quad F_1(x_j) \le s(x_j) \le F_2(x_j), \quad j = 0, ..., N-2,$$
$$s(\alpha) = x_0, \quad s(\beta) = x_1.$$

That simply could be solved using *NLPSolve* command in MAPLE software.

Example 3.1. Consider the given SPP in[1] with one lower barrier

$$Min \qquad J = \int_{-5/4}^{5/4} \sqrt{1+x'^2(t)}dt,$$

$$\textit{subject to} \quad x(t) \ge 1-t^2, \quad \forall_t \in \left[-\frac{5}{4}, \frac{5}{4}\right], \quad x\left(-\frac{5}{4}\right) = 0, \quad x\left(\frac{5}{4}\right) = 0.$$

Where the optimal value of the objective functional is $J^* = 3.26911$ and the exact solution of this SPP is as follows:

$$x^*(t) = \begin{cases} t + \frac{5}{4}, & -\frac{5}{4} \le t \le -\frac{1}{2}, \\ 1 - t^2, & -\frac{1}{2} \le t \le \frac{1}{2}, \\ -t + \frac{5}{4}, & \frac{1}{2} \le t \le \frac{5}{4}. \end{cases} \xrightarrow{t=\frac{5}{4}k} x^*(k) = \begin{cases} \frac{5}{4}k + \frac{5}{4}, & -1 \le k \le -\frac{4}{10}, \\ 1 - (\frac{5}{4}k)^2, & -\frac{4}{10} \le k \le \frac{4}{10}, \\ -\frac{5}{4}k + \frac{5}{4}, & \frac{4}{10} \le k \le 1. \end{cases}$$

Tables and Figures

TABLE 1. Numerical solution result of J^* for Example 3.1.

Legendre collocation (LC) method [1]		Present method	
N	J_N^*	N	J_N^*
4	3.26019	6	3.26969
8	3.26492	9	3.26861
18	3.26920	12	3.26887
32	**3.26911**	**13**	**3.26911**

TABLE 2. Numerical comparison for $x^N(k_i)$ for Example 3.1.

k_i	exact solution $x^*(k_i)$	LC method [1] N=32	LC method [1] $\lvert Error \rvert$	Present method N=22	Present method $\lvert Error \rvert$
-1.0	0.0000	0.0000	0.0000	0.0000	0.0000
-0.8	0.2500	0.2503	0.0003	0.2500	0.0000
-0.6	0.5000	0.5007	0.0007	0.4999	0.0001
-0.4	0.7500	0.7505	0.0005	0.7504	0.0004
-0.2	0.9375	0.9375	0.0000	0.9374	0.0001
0.0	1.0000	1.0000	0.0000	1.0003	0.0003
0.2	0.9375	0.9375	0.0000	0.9374	0.0001
0.4	0.7500	0.7505	0.0005	0.7509	0.0009
0.6	0.5000	0.5007	0.0007	0.4998	0.0002
0.8	0.2500	0.2503	0.0003	0.2498	0.0002
1.0	0.0000	0.0000	0.0000	0.0000	0.0000
Total error			0.0030		0.0023

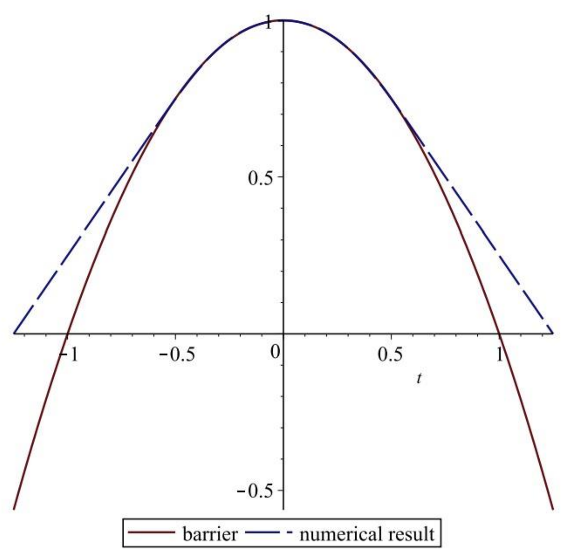

FIGURE 1. Optimal solution history of Example 3.1. with one lower boundary barrier for $N = 22$.